

Bound entanglement for continuous variables is a rare phenomenon

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We discuss the notion of bound entanglement (BE) for continuous variables (CV). We show that the set of non-distillable states (NDS) for CV is nowhere dense in the set of all states, i.e., the states of infinite-dimensional bipartite systems are generically distillable. This automatically implies that the sets of separable states, entangled states with positive partial transpose, and bound entangled states are also nowhere dense in the set of all states. All these properties significantly distinguish quantum CV systems from the spin like ones. The aspects of the definition of BE for CV is also analysed, especially in context of Schmidt numbers theory. In particular the main result is generalised by means of arbitrary Schmidt number and single copy regime.

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I. INTRODUCTION

Bound entanglement [1] is the entanglement which cannot be distilled, i.e. no pure state entanglement can be obtained from it by means of local operations and classical communication (LOCC) [2]. So far, it has been studied mainly for spin like systems. These studies has allowed to discover many interesting properties of bound entanglement, for both bipartite [3], and multiparticle systems [4]. Recently, much attention has been devoted continuous variable (CV) systems (c.f. [5]). Bound entanglement has also been considered for continuous variables (CV), and the first nontrivial examples of BES for CV have been constructed [6] (see also [7]). Once we have some examples of BES for CV, it is interesting to ask how frequent is the phenomenon of bound entanglement, i. e. how many states of that kind are in set of all CV states?

The question of "how many quantum states having some interesting property are there?" is very natural. In the context of entanglement it was first considered in Ref. [8], where the problem of the volume of the subset of separable (non-entangled) states in the set of all bipartite states of spin systems was considered. Numerical evidence has shown that the volume of the set of separable states approaches zero when the size of the spin goes to infinity. It was also shown that for any finite spin system the volume of separable states is nonzero due to the existence of a separable neighborhood, i.e. an

open ball of separable states in the vicinity of the maximally mixed state in arbitrary dimension. Further first analytical bounds on the size of neighbourhood have been provided [9]. All this raised a series of questions concerning the interpretation of experiments of quantum computing based on high temperature NMR; many interesting analyses have been performed in this context [10,11].

The question of the "size" of the set representing separable states has been recently answered [12] for CV; it has been show that for bipartite states this subset is nowhere dense (relative to the trace-norm topology). This implies that this set does not contain any open ball and also that CV states are generically non-separable. On the other hand there exists another subset that is of interest in the context of entanglement. This is the subset of non-distillable states (NDS), i.e. states that cannot be distilled. This subset contains the separable states, and therefore it might well be that an appreciable fraction of all states are in such a subset. In this paper we show that this is not the case; that is, the subset of NDS is nowhere dense in the set of bipartite states of CV. We present two different proofs of this fact. One uses the uniform topology, and the other one the trace-norm topology.

We also perform analysis how one can relax conditions of NDS in context of CV in comparison with the standard definition, and prove stronger version of the main result with help of Schmidt numbers theory [13] and single copy regime (see [15,14]).

There are several results which follow from our proofs. In particular, since the subset of NDS contains the subset of BES, we have that that subset is also nowhere dense. The same thing occurs with the subset of states with positive partial transpose (PPT) [1] and therefore with with those PPT states that are entangled. Moreover, since the subset of separable states is also contained in the one of NDS, our results include the ones given in Ref. [12].

II. BOUND ENTANGLED STATES FOR CONTINUOUS VARIABLES

The main subject of this paper is the question about whether generic CV states are non-distillable. We present below the answer to this question: the subset

of NDS is nowhere dense. In this section we first discuss the definition of NDS. Then, we present two proofs of our result. First, it will be proven using the uniform topology and exploiting the fact that any density operator can be considered as the limit of a sequence of density operators defined on a finite support. Then, following the approach of Ref. [12] we shall prove the general statement that any proper closed subset of bipartite states which is invariant under local transformations is nowhere dense (in the trace norm topology). This generalizes the results of Ref. [12] and, as we shall see, together with the fact that the set of NDS is closed, proves our claims.

A. Bound entangled states

Let us denote by M set of all density operators acting on $H_A \otimes H_B$, where $H_A \cong H_B \cong L^2(R)$. Let us consider a density operator $\rho \in M$. According to Ref. [1], ρ is distillable [1] iff there exists some finite $n \in \mathbb{N}$, two rank two projectors P_A, P_B acting on H_A, H_B , respectively, such that

$$((P_A \otimes P_B)\rho^{\otimes n}(P_A \otimes P_B))^{T_B} \not\geq 0; \quad (1)$$

otherwise, ρ is non-distillable. We call D and N the set of all distillable and non-distillable density operators, respectively. Physically, this definition tells us that a state is distillable iff out of a sufficiently large number of copies we can obtain by local operations two qubits which are entangled. The reason for that is clear. First, given the fact that one can distill maximally entangled states out of all entangled states of qubits, this means that if the above condition is true, we can always distill maximally entangled qubit states out of the original state ρ . Second, if the above condition is not fulfilled for any n , then we will not be able to produce (asymptotically) any qubit maximally entangled state by using local operations alone.

In the following we will reexpress Eq. (1) as follows:

$$\epsilon \equiv \langle \Psi | ((P_A \otimes P_B)\rho^{\otimes n}(P_A \otimes P_B))^{T_B} | \Psi \rangle < 0 \quad (2)$$

for some $|\Psi\rangle \in H_A \otimes H_B$. Without loosing generality we can take $|\Psi\rangle = P_A \otimes P_B^{T_B} |\Psi\rangle$, i.e. $|\Psi\rangle$ belongs to the 2×2 subspace determined by $P_A, P_B^{T_B}$. It is easy to see then that

$$(|\Psi\rangle\langle\Psi|)^{T_B} = \sum_{k=1}^4 \lambda_k |\phi_k\rangle\langle\phi_k|, \quad (3)$$

where $-1/2 \leq \lambda_k \leq 1$, i.e. $|\lambda_k| \leq 1$ and the $|\phi_k\rangle$ are also in the same 2×2 subspace.

B. Bound entangled states for CV are nowhere dense: Proof I

In this subsection we show that the set N of NDS states is *nowhere dense* in the set of all states M . We will first

recall some definitions. A subset A of a topological space X is nowhere dense if its closure contains no open set. Note that if A is already closed then it is nowhere dense iff it contains no open set. For example, the subset of integer numbers Z in R (with the topology induced by the absolute value metric) is nowhere dense since it is already closed and no neighborhood of any integer contains only integers.

In this subsection we will use the uniform topology for operators, which is the one derived from the operator norm. First we will show that N is closed with that topology by proving that its complement, D , is open. Then, we will show that D is dense in the set of all density operators. From this last point it follows that N (equivalently, its closure) contains no open set and therefore it is nowhere dense.

In order to show that D is open, let us consider some $\rho \in D$. According to the definition of D , there exists some finite integer n , and a Schmidt rank two state $|\Psi\rangle \in H_A \otimes H_B$ such that Eq. (2) is fulfilled with (3). Let us consider an open ball $B_\eta(\rho) = \{\rho', ||\rho' - \rho|| < \eta\}$. We will show that for $\eta < |\epsilon|/4n$, $B_\eta(\rho) \subset D$.

To this aim we argue that

$$\begin{aligned} & |\langle \Psi | ((P_A \otimes P_B)(\rho^{\otimes n} - (\rho')^{\otimes n})(P_A \otimes P_B))^{T_B} | \Psi \rangle| \\ &= \left| \sum_{k=1}^4 \lambda_k \langle \phi_k | \rho^{\otimes n} - (\rho')^{\otimes n} | \phi_k \rangle \right| \\ &\leq 4 ||\rho^{\otimes n} - (\rho')^{\otimes n}|| \leq 4\eta n. \end{aligned} \quad (4)$$

The latter inequality can be proven by induction, using the identity

$$\begin{aligned} \rho^{\otimes n} - (\rho')^{\otimes n} &= \frac{1}{2}(\rho^{\otimes n-1} + (\rho')^{\otimes n-1}) \otimes (\rho - \rho') \\ &\quad + \frac{1}{2}(\rho^{\otimes n-1} - (\rho')^{\otimes n-1}) \otimes (\rho + \rho'), \end{aligned} \quad (5)$$

and the fact that both $||\rho||$ and $||\rho'||$ are smaller than one. Thus, if $\eta < |\epsilon|/4n$, we see that for any $\rho' \in B_\eta(\rho)$,

$$\langle \Psi | ((P_A \otimes P_B)(\rho')^{\otimes n}(P_A \otimes P_B))^{T_B} | \Psi \rangle < 0, \quad (6)$$

ergo ρ' is distillable.

Now we show that D is dense in M . To this aim we observe that for any $\rho \in M$ we can always find a sequence $\{\rho_n\}_{n=0}^\infty$, with $\rho_n \in D$ such that $\rho_n \xrightarrow{u} \rho$. We consider the spectral decomposition of ρ as

$$\rho = \sum_{n=1}^\infty p_n |\Psi_n\rangle\langle\Psi_n|, \quad (7)$$

where we have chosen $p_1 \geq p_2, \dots$. Note that since ρ is a trace class operator, the sequence p_n converges monotonically to zero. On the other hand, we can write the Schmidt decomposition of each $|\Psi_n\rangle$ as

$$|\Psi_n\rangle = \sum_{k=1}^{\infty} \sqrt{\lambda_{n,k}} |u_{n,k}, v_{n,k}\rangle, \quad (8)$$

where again we have chosen $\lambda_{n,k} \geq \lambda_{n,k+1} \geq 0$ and $\lambda_{n,k}$ converges monotonically to zero as $k \rightarrow \infty$. Now, we define

$$\tilde{\rho}_N \equiv \sum_{n=1}^N p_n |\Psi_{N,n}\rangle \langle \Psi_{N,n}|, \quad (9)$$

where

$$|\Psi_{N,n}\rangle = \sum_{k=1}^N \sqrt{\lambda_{n,k}} |u_{n,k}, v_{n,k}\rangle. \quad (10)$$

It is clear that $\tilde{\rho}_N$ is supported on $H_A^N \otimes H_B^N$, where both H_A^N have finite dimension. Thus, we can always find two pairs of orthogonal vectors $|a_{1,2}^N\rangle \in H_A \ominus H_A^N$ and $|b_{1,2}^N\rangle \in H_B \ominus H_B^N$. Let us define $|\Phi^N\rangle = (|a_1, b_1\rangle + |a_2, b_2\rangle)/\sqrt{2}$ and

$$\rho_N = K_N (\tilde{\rho}_N + \frac{1}{N} |\Phi^N\rangle \langle \Phi^N|), \quad (11)$$

where K_N is a normalization constant. It is clear that $\rho_N \xrightarrow{N \rightarrow \infty} \rho$. On the other hand, defining

$$P_A^N = |a_1\rangle \langle a_1| + |a_2\rangle \langle a_2|, \quad (12a)$$

$$P_B^N = |b_1\rangle \langle b_1| + |b_2\rangle \langle b_2| \quad (12b)$$

and taking $n = 1$ it is clear that $\rho_N \in D$. This completes the proof. \square

Obviously the fact that the set of NDS states for CV is nowhere dense, implies that the contained in it sets of BES and PPT states are also nowhere dense. One can, however, prove the latter directly using the method used above. The only difference would be that the state $|\Psi\rangle$ which in above proof belongs to a $2 \otimes 2$ subspace has to be substituted by a general vector $|\Psi\rangle$ of arbitrary Schmidt rank, but at the same time there is no need to consider n -fold tensor products, since the PPT property of ρ is maintained for arbitrary number of its copies. One has to use, however, the property of the partially transposed projector $\|(|\Psi\rangle \langle \Psi|)^{T_B}\| \leq 1$.

C. Bound entangled states for CV are nowhere dense: Proof II

In order to demonstrate the statement of the previous section in another way, we first need to recall notions of several necessary tools, which have been used in Ref. [12]. Let $\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2)$ stand for bounded operators on Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ describing our bipartite system. We assume that at least one of the subsystems is described by CV, and hence it has the infinite dimension.

Now, consider a third auxiliary system described by \mathcal{H}_3 . It is convenient to describe all states in $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ as reduced states of some *pure* states in the extended space $\mathcal{H} \otimes \mathcal{H}_3$. If we have a pure state $|v_{123}\rangle \langle v_{123}|$ in the extended space, then the reduced state $\text{Tr}_3(|v_{123}\rangle \langle v_{123}|)$ is denoted by ϱ_{12} . Let us denote by \mathcal{T} the set of all states on $\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2)$. This is a set of unit trace operators with nonnegative spectrum. We shall also endow this set with the norm topology $\|\cdot\|_T$, $\|A\|_T \equiv \text{Tr}(\sqrt{A^\dagger A})$. Now, one defines [12] the map Φ from the unit sphere \mathcal{S} representing all wavefunctions form $\mathcal{H} \otimes \mathcal{H}_3$ to the set of states \mathcal{T} in the following way :

$$\Phi(|v_{123}\rangle) = \text{Tr}_3(|v_{123}\rangle \langle v_{123}|) = \varrho_{12}. \quad (13)$$

The map $\Phi : \mathcal{S} \rightarrow \mathcal{T}$ is *continuous* (in the norm $\|\cdot\|_T$) and *onto*. In particular it maps dense subsets onto dense subsets (see [12] for explanation).

Consider the set \mathcal{X} of all vectors $u_{123} = A \otimes I \otimes I(v_{123})$ for all $A \in \mathcal{B}(\mathcal{H}_1)$. The vector v_{123} is called *1-cyclic* (see [12]) if the closure of \mathcal{X} in the norm $\|\cdot\|_T$ turns out to be the whole space $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$. The physical interpretation of 1-cyclic vectors in both finite dimensional, as well as in the CV case, is that those are the vectors which have maximal possible Schmidt rank. Note, that according to Lemma 2 of Ref. [25] they form a dense set in $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$.

Now we consider the following simple

Observation 1.- Let the set \mathcal{ND} be (i) a proper closed (in $\|\cdot\|_T$ norm) subset of the set of states \mathcal{T} which is (ii) invariant under the operations $A \otimes I$. Then any vector v_{123} satisfying $\Phi(v_{123}) \in \mathcal{ND}$ cannot be 1-cyclic and \mathcal{ND} is nowhere dense in \mathcal{T} .

The above observation is a natural generalization of the Lemma 1 of Ref. [12]. To show this, consider such vector v that its “reduction” $\Phi(v)$ belongs to \mathcal{ND} , and take any vector

$$v' = A \otimes I \otimes I(v), \quad (14)$$

defined for arbitrary A , such that $\|v'\| = 1$. We shall show first that $\Phi(v')$ also belongs to \mathcal{ND} . Indeed (see [12]) we have $\Phi(v') = A \otimes I \Phi(v) A^\dagger \otimes I$ and (because the norm of v' is one) the trace $\Phi(v')$ is one. But, because the set \mathcal{ND} is closed under the operation $A \otimes I(\cdot) A^\dagger \otimes I$, we see that $\Phi(v')$ still belongs to the set.

Now suppose that v were 1-cyclic. Then, that the set \mathcal{M} of all vectors v' would be dense in the unit sphere \mathcal{S} of all normalized vectors belonging to $\mathcal{H} \otimes \mathcal{H}_3$. As the map Φ is continuous and onto, it certainly would map \mathcal{M} onto some new set denoted by $\Phi(\mathcal{M})$, which would be dense in set of all bipartite states \mathcal{T} . Thus closure of $\Phi(\mathcal{M})$ must have give all \mathcal{T} . But, on the other hand any element of $\Phi(\mathcal{M})$ (which is defined as $\Phi(v')$ for some vector v' of the form (14)) belongs to \mathcal{ND} . As the latter is closed, the closure of $\Phi(\mathcal{M})$ would have to be a subset of \mathcal{ND} . But \mathcal{ND} was supposed to be closed and strictly

smaller than the set \mathcal{T} , so the closure of $\Phi(\mathcal{M})$ cannot be equal to \mathcal{T} . This gives the required contradiction. The above reasoning follows the lines of the proof of Ref. [12]. The only difference is that instead of the specific set of separable states considered there, here we have considered an abstract set \mathcal{ND} , which has some special properties. Note, that the assumptions of Observation 1 and the fact that the 1-cyclic vectors form a dense set in \mathcal{S} imply that the closed set \mathcal{ND} is nowhere dense. If it had contained an open set, then, following continuity of Φ this open set would have had to be an image of open subset of $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$, which would have had to contain a ball, and thus a 1-cyclic vector. Now, to show that the set of NDS states is nowhere dense we have to show that it is (i) invariant under local operations of the type $A \otimes I$, (ii) closed in the trace norm $\|\cdot\|_T$. The first property (i) is immediate, since a NDS cannot be converted into a free entangled state by means of local operations. The second one is not so obvious for continuous variables, but it follows from the results of the previous subsection. We thus have:

Observation 2.- The property of non-distillability is invariant under the one side local action $A \otimes I(\cdot)A^\dagger \otimes I$.

The proof is simple - the arguments of Ref. [1] can be applied (see also [6]) to show that any local separable superoperator cannot cause that the state loses the non-distillability property.

Observation 3.- The set of all NDS is closed in the norm $\|\cdot\|_T$.

To prove the closeness of the set of NDS, we prove that its complement, i.e. the set of distillable states D is open in the trace norm. To this aim we repeat the arguments of subsection A and consider some $\rho \in D$, for which there exists some finite integer number n , P_A, P_B , rank two projectors acting on H_A, H_B , and a rank two vector $|\Psi\rangle \in H_A \otimes H_B$ such that Eq. (2) is fulfilled. We consider now an open ball in the trace norm, i.e. $\tilde{B}_\eta(\rho) = \{\rho', \|\rho' - \rho\|_T < \eta\}$. Note that if $\rho' \in \tilde{B}_\eta(\rho)$ then the operator norm fulfills $\|\rho' - \rho\| \leq \|\rho' - \rho\|_T < \eta$ [16]. Using the same argument as before we show that for $\eta < |\epsilon|/4n$, $\tilde{B}_\eta(\rho) \subset D$, which completes the proof. \square

Combining the Observations 1.-3. we see that the set of NDS states is nowhere dense in the trace norm, which implies the same property for the BES, PPT states, and separable states.

III. ANALYSIS

As mentioned in the introduction, non-trivial BES for continuous variables (CV) have been discovered. In this subsection we discuss some of the details regarding these states, as well as whether entangled states in CV with infinite Schmidt number represent are generic in the set of entangled states.

A. Continuous variable bound entangled states

The construction of non-trivial BES for CV systems [6] was based on an idea similar to the one used for spin systems, for which it has been proven that any entangled state with positive partial transpose [17] cannot be distilled [1]. The crucial element of the construction was to create the state in such a way that it cannot be obtained simply by embedding a bound entangled state in a finite dimensional Hilbert space into the Hilbert space of CV.

The particular example ϱ of CV BES proposed by us was first of all assumed to satisfy the condition that its partial transpose ϱ^{T_B} , defined as

$$\varrho_{m\mu, n\nu}^{T_B} \equiv \langle m, \mu | \varrho^{T_B} | n, \nu \rangle = \varrho_{m\nu, n\mu}, \quad (15)$$

has a nonnegative spectrum. Such requirement was, however, not sufficient, as one could invent the following "trivial" example of a PPT entangled states for CV [6]

$$\tilde{\sigma} = \bigoplus_{n=1}^{\infty} p_n \sigma_n. \quad (16)$$

The above state is build from infinitely many "copies" of the same $3 \otimes 3$ BES σ labeled by σ_n . Each of σ_n has the matrix elements of the original σ , but in the basis $S_n = \{|i, j\rangle\}_{i,j=3n}^{3n+3}$. Here $\{p_i\}_{i=1}^{\infty}$ is an infinite sequence of nonzero probabilities, $\sum_{i=1}^{\infty} p_i = 1$. The bound entanglement of the CV state $\tilde{\sigma}$ is in a certain sense spurious, as it can in principle be reversibly converted by means of local operations and classical communication into the $3 \otimes 3$ entanglement.

One could easily construct another example, similar to the one above, with σ_n acting in $k_n \otimes k_n$ Hilbert space with $k_n \rightarrow \infty$, and $\tilde{\sigma}$ being block diagonal as in (16). This example is much more interesting as far as CV are concerned, because it can not be reversibly converted into any state of fixed spin. Thus to some extent it might be regarded as generic BE. However, such a state would still be a mixture of "locally orthogonal" spin states, which does not exploit fully the CV Hilbert space structure, i.e. infinite dimension fully. In fact, if such CV BES were produced by a random mixture, they could be easily "decoupled" by local projective measurements and classical communication.

Thus we propose to define generic BES for CV in a stronger way, namely as the states from which no pure entanglement can be distilled, and they cannot be represented by the states of the above sort.

This is a somewhat phenomenological definition, but it implies to single out some required properties of the

¹Subsequently we shall denote by $n \otimes n$ states the states of quantum systems defined on the Hilbert space $\mathcal{H} = \mathcal{C}^n \otimes \mathcal{C}^n$. The space will be sometimes called " $n \otimes n$ space".

generic CV BES. The first nontrivial examples of the generic CV BES, presented in Ref. [6], fulfill those requirements. These states have the form:

$$\varrho \propto |\Psi\rangle\langle\Psi| + \sum_{n=1}^{\infty} \sum_{m>n}^{\infty} |\Psi_{mn}\rangle\langle\Psi_{mn}|, \quad (17)$$

with the following definitions of the symbols: $|\Psi\rangle = \sum_{n=1}^{\infty} a_n |n, n\rangle$, $|\Psi\rangle \in \mathcal{H} = l^2(\mathcal{C}) \otimes l^2(\mathcal{C})$ with the finite norm $\|\Psi\|^2 = \sum_{n=1}^{\infty} |a_n|^2 = q < \infty$, and vectors

$$|\Psi_{mn}\rangle = c_m a_n |n, m\rangle + (c_m)^{-1} a_m |m, n\rangle, \quad (18)$$

for $n < m$ with (in general) complex a_n and c_n , such that (i) $0 < |c_{n+1}| < |c_n| < 1$, (ii) $\sum_{n=1}^{\infty} \sum_{m>n}^{\infty} \|\Psi_{mn}\|^2$ is finite. The latter condition can be achieved for example by setting $a_n = a^n$, $c_n = c^n$, for some $0 < a < c < 1$, see [6]. Physically, the vector $|\Psi\rangle$, when normalized, may describe a state of two modes of the quantized electromagnetic field, or more generally two harmonic oscillators. The state (17) has the following properties: (i) it is bound entangled, as it has the PPT property (i. e. it has the positive partial transpose); (ii) it is not a simple “direct sum” of finite spin BES in a sense of the “spurious” examples discussed above (Eq. (16)).

Recently considerable attention has been devoted to the so called Gaussian states. In systems of two harmonic oscillator modes (one of Alice, one of Bob), i.e. in the, so called, Gaussian 1×1 case, it has been shown that no bound entanglement exists – such Gaussian states are either separable [18,19], or distillable [20]. In another words, in this case PPT property is a necessary and sufficient condition for separability, and non-distillability. This result can be extended to the case $1 \times N$. Soon after realizing this facts Werner and Wolf have found an example of a Gaussian BES with PPT property [7]. This result has been achieved by considering first covariance matrices of Gaussian states and their null subspaces. It was noted that the Gaussian state is separable, iff its covariance matrix can be minorized by some block diagonal covariance matrix. Second, the characterization of PPT states in terms of covariance matrix has been found. The BES has been constructed using an elegant explicit construction, performed using the analysis of the range and the “subtraction method” first developed for spin systems in Refs. [21–24]. In the terminology of Refs. [21–24] the states found in Ref. [7] are examples of the so called “edge states”. The approach of Ref. [7] can be used further to analyze multiparticle entanglement. In particular, one can try to “split” the covariance matrix of $n \times n$ state in a way to get $m \times m \times m$ state with some bound entanglement properties. Indeed, we have recently managed to solve the separability problem for the case of tripartite system with one mode per each party [25]. The result of Werner and Wolf appeared first a little surprising in the view of Refs. [18–20]. Recently, some of

us have been able to clarify this and solve ultimately the separability [26] and distillability [27] problems for Gaussian states of two parties sharing arbitrary number of modes. While the PPT property remains a valid necessary and sufficient condition for non-distillability, the separability criterion has a complex form of a nonlinear map for covariance matrices.

Finally, it is worth mentioning that it is not known yet whether there exist BES which do not have PPT, even though there is a strong indication of this fact [28]. If this were finally true, this would have important implications in the context of distillation [29], since it may well happen that by mixing two NDS one obtains a distillable one.

B. Question of genericity: the structural point of view

There is one open question whether the given CV entanglement represents a *generic* entanglement in the sense that it has infinite Schmidt number (see [13]), i. e. whether it is the limit of matrices whose Schmidt number goes to infinity. This means that, in principle, in order to generate the state, one would have to be able to generate the states of arbitrary Schmidt rank. However, some of the CV BES states similar to “spurious” ones could have also this property - if a finite dimensional $n \otimes n$ PPT states with Schmidt rank of order $O(n^\alpha)$ with some $0 < \alpha \leq 1$ existed, then we could put in the expression (16) the $k_n \otimes k_n$ states σ_n with the rank $O(k_n^\alpha)$ where say $k_{n+1} = 2(\sum_{i=1}^n k_i)$. Thus, we see that in order to describe the generic CV entangled states it seems reasonable to require the stronger version the notion of infinite Schmidt number. Intuitively, it should mean that the pure states with infinite Schmidt rank are necessarily involved in the mixed state representation. One possible definition would be that a generic CV state with infinite Schmidt rank should be necessarily of the form $\varrho = \sum_i p_i |\Psi_i\rangle\langle\Psi_i|$, with $|\Psi_i\rangle$ not necessarily orthogonal, but with *at least one* $|\Psi_i\rangle$ of infinite rank. Such states obviously exist – take for instance one pure state of infinite Schmidt rank, or a convex combination of two such states. However, in the above definition the precise notion of the decomposition in the CV case in the sense of Ref. [30] has to be specified. Another possible definition (which seems to be significantly weaker) would be to require the generic CV state to be the limit of $n \otimes n$ states of Schmidt rank n^α for some $0 < \alpha \leq 1$.

Concerning BES – we do not know whether there exists any BES for CV with PPT property, having at the same time the feature of being a generic CV state, whatever it would mean. It is worth stressing at this point that according to the results of Ref. [31], PPT entangled state in $n \otimes n$ space are expected to have Schmidt number smaller than n . In fact, in the Appendix A, we present the arguments analogous to those used in Ref. [31] that the

typical PPT bound entangled states in $n \otimes n$ space either have the Schmidt number of order $O(1)$, or their partial transpose have this property. It is possible, however, that the recently introduced Gaussian bound entangled states [7,26] satisfy all requirements as far as the CV genericity is concerned. It would thus be interesting to analyze the Schmidt number of those states.

C. Question of genericity: distillation point of view

In former section we have dealt with question of genericity of CV state as far as *the structure of the state is concerned*. Bound (nondistillable) entanglement is directly related to distillation procedures. It is important to address the question from different point of view i. e. analysing the output of distillation procedure from the point of view of genericity.

The present result of section II clearly shows that NDS in the sense of standard definition (that no entanglement can be distilled from given state) is nowhere dense in set of CV states. This is an important theorem generalising previous results. However in the classical definition of NDS treats both finite and infinite dimensional entanglement nondistillable. For sake of many applications the next step of study which would operationally distinguish those two quantities would be desirable.

In such approach “fully CV NDS ” would be all those states that do not allow for distillation of *infinite* pure entanglement (whatever it means). This significantly increases the set of states that are interpreted as bound entangled.

As we shall see below it is more complicated issue. We will not give definite answers here. However further methods of investigation will be suggested.

Again, as in previous section, one of proposed definitions could be the following:

A. The state ϱ represents “fully CV free (nondistillable)” entanglement if and only if it is possible (impossible) to distill nonzero amount of pure states with infinite Schmidt rank from state ϱ .

Nonzero amount is here understood in sense of usual distillation yield (i. e. as a nonzero amount of pairs). Note that to qualify distilled entanglement in finite dimensions the condition of asymptotic approaching the maximally entangled state $|\Psi_+\rangle = \frac{1}{\sqrt{m}} \sum_{i=1}^m |e_i, e_i\rangle$ was required. It is known however that there is no maximally entangled states of infinite Schmidt rank. Thus in place of Ψ_+ one would have probably use some fixed pure state Ψ_∞ having the reduced density matrix nonsingular or at least of infinite rank (this is equivalent to the infinite Schmidt rank of Ψ_∞).

Another interesting (weaker) definition would be more in spirit of Ref. [6] where increasing sequences of finite Schmidt rank were used. Namely one can propose:

B. The state ϱ represents CV free (nondistillable) entangled if and only if it is possible (impossible) to distill nonzero amount $\eta_p > 0$ of p -Schmidt rank states with $\limsup_p \eta_p > 0$.

The main difficulty dealing with Schmidt rank in those definitions is that the operational methods of its detection in context of CV are not enough developed.

For example it is not known whether the proposal B above is equivalent to the following generalisation of the “two-qubit subspace” (see sec. II A):

the state is fully free (bound) entangled iff there is (no) n and the family of bilocal filters $A_p \otimes B_p$ such that the new n -copy states

$$\varrho'_p = A_p \otimes B_p \varrho^{\otimes n} A_p^\dagger \otimes B_p / (A_p \otimes B_p \varrho^{\otimes n} A_p^\dagger \otimes B_p) \quad (19)$$

violate the p -Schmidt rank test via positive map i. e. $[\mathbb{I} \otimes \Lambda_p](\varrho'_p)$ is not positive matrix. The map $\Lambda_p(X) \equiv \text{Tr}(X)I - (p-1)^{-1}X$ is $p-1$ -positive but not p -positive and was used to detect p -Schmidt rank of isotropic entangled states [13].

Dealing with the state (19) is not easy because even in the case of finite dimensions the possibility of asymptotically singular denominator in formulas like (19) leads to suprising effects (see [15]). Putting $A_p \otimes B_p$ equal identity we shall make the results of section II slightly stronger and in the above spirit.

IV. BES FOR CV ARE NOWHERE DENSE: GENERALISATION INVOLVING SCHMIDT NUMBERS

Here, using the trace norm topology from sec. II. B we shall prove the stronger version of the main result of sect. II. Suppose for moment that as a “fully CV free entanglement” we shall treat *only* very special CV states. Namely only those form which it is possible to produce p -Schmidt rank state (with p fixed but *arbitrary* high) from *a single copy* by means of special LOCC protocol given in Appendix B. The protocol is a natural generalisation of the protocol utilising reduction criterion [14].

Those special states form the set, say D_p^1 (1 stands for “single copy” and p for Schmidt rank). This set is significantly smaller than the one formed by the classical definition of free entangled states (see sec. II. A.). If as “fully CV free states” one treats the set D_p^1 (which still seems to contain too much states, c. f. discussion of sec. III.B, but this is for didactic purposes) than one enlarges the set of what is understood as “fully CV bound” or “fully CV nondistillable”.

Below we shall see that though the latter is larger it is still nowhere dense. To have it we need to prove that D_p^1 is (i) open and (ii) dense in set of all density operators. Any state $\varrho \in D_p^1$ satisfies by the very definition (see Appendix B) the inequality:

$$\langle \Psi | [\mathbb{I} \otimes \Lambda_p](\varrho) | \Psi \rangle = \epsilon < 0. \quad (20)$$

Now suppose that $\varrho' \in \tilde{B}_\eta(\varrho) = \{\varrho', \|\varrho - \varrho'\|_T < \eta\}$. Then

$$\begin{aligned} |\langle \Psi | [\mathbb{I} \otimes \Lambda_p](\varrho - \varrho') | \Psi \rangle| &= |\langle \Psi | (\varrho_A \otimes I - \varrho'_A \otimes I \\ &+ (p-1)^{-1}(\varrho - \varrho')) | \Psi \rangle| \leq |Tr[\varrho_\Psi(\varrho_A - \varrho'_A)]| + \\ |\langle \Psi | (p-1)^{-1}(\varrho - \varrho') | \Psi \rangle| &\leq Tr[\varrho_A^\Psi \varrho_A - \varrho'_A] + \\ (p-1)^{-1}|\langle \Psi | \varrho - \varrho' | \Psi \rangle| &\leq \eta(1 + (p-1)^{-1}) \end{aligned} \quad (21)$$

where ϱ_A^Ψ is reduced density matrix of pure state $|\Psi\rangle\langle\Psi|$.

Where we have used two properties (i) $\|\varrho_A - \varrho'_A\|_T \leq \eta$ because partial trace is tracepreserving completely positive map which does not increase the trace norm $\|A\|_T$; (ii) $\langle \phi | A | \phi \rangle \leq \|A\| \leq \|A\|_T$ for positive A where $\|A\|$ stands for operator norm as in sect. II.B. Thus, for $\eta < \frac{|\epsilon|}{(1+(p-1)^{-1})}$, any $\varrho' \in \tilde{B}_\eta(\varrho)$ satisfies $\langle \Psi | [\mathbb{I} \otimes \Lambda_p](\varrho') | \Psi \rangle < 0$ i. e. *ergo* belongs to D_p^1 . This implies that the latter is open. Now to prove that D_p^1 is dense one can repeat the argument of sec. II.A with $|\Psi_N\rangle \in (H_A \ominus H_A^N) \otimes (H_B \ominus H_B^N)$ being maximally entangled pure state of Schmidt rank p : $|\Psi_N\rangle = \frac{1}{\sqrt{p}} \sum_{i=1}^p |e_i, f_i\rangle$ (the only difference is that the resulting state should be shown to satisfy (20) which is easy to see). Thus D_p^1 is dense and open so its complement is nowhere dense in set of all states which completes the proof. Remarkable that the line of the proof remains completely correct for “uniform” assumption (i. e. 20) satisfied with one ϵ for *all* natural p some $|\Psi\rangle = |\Psi(p)\rangle$) but than the assumption itself can be easily shown to be *false* in the sense that no state can satisfy it.

V. CONCLUSIONS

In this paper we have considered non-distillable states for continuous variables. In the main part of the paper we have proven that the subset of non-distillable states is nowhere dense in the set of all CV states. This is a much stronger result than the recent one by Clifton and Halvorson [12], which prove the same result for the set of separable states, since that one is contained in the set of NDS. Moreover, our results imply that the subsets of BES and PPT states are also nowhere dense. Thus, generic CV states are distillable. We have also presented some examples of BES and discussed their genericity from the point of view of CV and their Schmidt number. In the Appendix A we have presented an evidence that all PPT BES in $n \otimes n$ systems either have Schmidt number smaller than $O(1)$, or their partial transposes have this property. Finally we have analysed the genericity of CV entanglement in context of Schmidt number. We studied the flexibility of the main result under the assumptions and proved more general result that nowhere sende is the set of all states from which it is impossible to produce

p -Schmidt rank state from a single copy in some (well defined) way. The latter involves single copy protocol (provided in Appendix B) being a generalisation of that obtained with help of reduction criterion. By providing some proposals of definitions of what can be treated as “fully CV” we have shown that further investigation of genericity in context of CV is desirable.

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APPENDIX A: SCHMIDT NUMBER OF PPT BES FOR $N \otimes N$ SYSTEMS

In this Appendix we essentially repeat the arguments used in the Ref. [31] to support the conjecture that in $3 \otimes 3$ systems all PPT BES have Schmidt number 2. We consider now the $n \otimes n$ case, with n large. Let $r(\rho)$ denotes the rank of ρ ; our aim is to present a strong evidence for the following conjecture:

Conjecture .- All PPT entangled states in $n \otimes n$ systems either have Schmidt number of the order of $O(1)$ or their partial transposes have this property.

Note, that this conjecture concerns for instance projections of the PPT BES (17) onto $n \otimes n$ spaces. We observe that

- It is enough to show the conjecture for the, so called, edge states [22–24], i.e. the PPT states δ such that there exist no product vector $|e, f\rangle$ in their range, such that $|e^*, f\rangle$ is in the range of the partially transposed operator δ^{TA} .
- Let $r(\rho)$ denotes the rank of ρ . It is likely that it is enough to prove the conjecture for the edge states of maximal ranks [22], i.e. those whose ranks fulfill $r(\delta) + r(\delta^{TA}) = 2n^2 - 2n + 1$. We expect that such states are dense in the set of all edge states. To show the latter statement, we consider an edge state $\tilde{\delta}$ which does not have maximal ranks. We can always add to it infinitesimal amount of projectors on product vectors destroying the edge property. The resulting state ρ would have more product states in its range, than the product states used to destroy the edge property. Subtracting projector on product states different from the latter ones, would typically allow to construct an edge state δ

with maximal ranks, which would be infinitesimally close to δ in any norm.

- Let $R(A)$, $K(A)$ denotes the range and kernel of A , respectively. The canonical form of a non-decomposable entanglement witness that detects the edge state δ is ([23,24], see also [32])

$$W = P + Q^{T_A} - \epsilon \mathbf{I}, \quad (\text{A1})$$

where the positive operators P, Q have their ranges $R(P) = K(\delta)$, $R(Q) = K(\delta^{T_A})$, and $\epsilon > 0$ is sufficiently small so that for any product vector $\langle e, f | W | e, f \rangle \geq 0$.

- If we can show that for any edge state with maximal ranks and any corresponding witness W detecting its entanglement, there exist a vector $|\psi^s\rangle$ of Schmidt number s such that $\langle \psi^s | W | \psi^s \rangle < 0$, then we would conclude that all edge states with maximal ranks, and thus all edge states, and thus all PPT entangled state have the Schmidt number $< s$. Equivalently, it is sufficient to show that $\langle \psi^s | W + \epsilon \mathbf{I} | \psi^s \rangle \leq 0$.

Let us therefore try to construct the desired vector $|\psi^s\rangle$ of Schmidt number s . In general such (unnormalized) vector will have a form

$$|\psi^s\rangle \propto \sum_{i=1}^s l_i |e_i, f_i\rangle, \quad (\text{A2})$$

where l_i are arbitrary complex coefficients for $i = 1, \dots, s$, and $|e_i, f_i\rangle$ are linearly independent product vectors for $i = 1, \dots, s$. Note, that the vector (A2) depends on s complex parameters l_i for $i = 1, \dots, s$, whereas each of the s vectors $|e_i\rangle, |f_i\rangle$ depends themselves of $n - 1$ relevant complex parameters.

Let $r(P) = k_1$, and $r(Q) = 2n - 1 - k_1$. Since we want to prove the conjecture either for the edge state δ , or for its partial transpose, without losing the generality, we may assume that $k_1 \geq 1$. We may then single out one projector out of P , and write $P = P_1 + |\Psi\rangle\langle\Psi|$, where $P_1 \geq 0$, $r(P_1) = r(P) - 1$, and $|\Psi\rangle$ is in the range of P . We can choose then $|e_i, f_i\rangle$ in such a way that $Q|e_i^*, f_i\rangle = 0$, and $P_1|e_i, f_i\rangle = 0$. These are effectively $2n - 2$ equations for vectors $|e_i, f_i\rangle$ which depend on $2n - 2$ parameters, so that we expect a finite, but quite large number of solutions (c.f. [22]). At the same time, $\langle \psi^s | Q | \psi^s \rangle$ will become a quadratic hermitian form of l_i 's with vanishing diagonal elements. Such a hermitian form has typically more than one dimensional subspace \mathcal{N} of negative eigenvalues for large s . But, one has to fulfill also the last equation implied by $\langle \psi^s | \Psi \rangle = 0$; this limits the values of l_i to a hyperplane, which should have at least one dimensional common subspace with the subspace of negative eigenvalues \mathcal{N} . This would prove that either the Schmidt number of δ or of δ^{T_A} is of the order of 1.

Note that for a given δ , if the presented construction can be shown to be successful for every witness of δ , then it provides a sufficient condition for the state δ to have the Schmidt number smaller than s .

APPENDIX B: PRODUCING STATE OF P -SCHMIDT RANK FROM SINGLE CV COPY

In this Appendix we briefly show how to produce (by means of local operations and classical communication - LOCC) the p -Schmidt rank state from any CV state violating the separability condition

$$[\mathbb{I} \otimes \Lambda_p](\varrho) \geq 0. \quad (\text{B1})$$

Following Ref. [20] this is further generalisation of the distillation protocol of Ref. [14] where the above criterion with $p = 2$ (called reduction criterion c. f. [34]) has been used. The separability tests of the form (B1) (which can be called $p - 1$ -reduction criteria) for $p > 2$ considered first in [13] are examples of general positive maps separability tests (see [33]).

Consider some CV state ϱ and suppose that there exists $|\Psi\rangle$ such that $\langle \Psi | [\mathbb{I} \otimes \Lambda_p](\varrho) | \Psi \rangle = \epsilon < 0$. Then following arguments of Ref. [20] we get that there must exist N such that for any $m > N$ the new $m \otimes m$ state ϱ_m produced from ϱ by projection onto the finitedimensional support of $\mathcal{H}_m \otimes \mathcal{H}_m$ satisfies

$$\langle \Psi | [\mathbb{I} \otimes \Lambda_p](\varrho_m) | \Psi \rangle \leq \frac{\epsilon}{2} < 0. \quad (\text{B2})$$

Now instead of $|\Psi\rangle$ we put $|\Psi'\rangle$ being a normalised projection of $|\Psi\rangle$ on support of ϱ_m . In this way we get the inequality identical to the one of [14] with the only difference that p was equal 2 there. This allows to repeat the reasoning of Ref. [14]: after the application of suitable local filtering and $U \otimes U^*$ twirling to ϱ_m one produces the $m \otimes m$ isotropic state $\varrho_{is} = (1 - q)\frac{\mathbf{I}}{m^2} + q|\Psi_+\rangle\langle\Psi_+|$ with the fidelity $F \equiv \langle \Psi_+ | \varrho_{is} | \Psi_+ \rangle > \frac{p-1}{m}$. But the latter implies (see [13]) that the final state (produced from initial ϱ by means of local operations and classical communication - LOCC) has the Schmidt number at least p . This concludes the reasoning. Note that further steps of recurrence protocol with generalised XOR ([14]) can be applied.

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